Derivatives of $x^n(x - 1)(x - a)$ with Rational Roots

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Abstract. Let $n \geq 3$ denote an integer and $a \neq 0, 1$ denote a rational number. For the family of polynomials $f(x) = x^n(x - 1)(x - a)$ with fixed value of $n$, we show that there exist infinitely many values of $a$ such that the first two derivatives of $f(x)$ have rational roots. We find two examples of $n$ and $a$ for which the first three derivatives of $f(x)$ have rational roots.

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1. Introduction

A polynomial $f(x)$ with rational coefficients is called rational derived if $f(x)$ and all of its derivatives have rational roots. The following example of a rational derived quartic was given by Carroll [4].

$$f(x) = (x + 167)^2(x - 141)(x - 193),$$
$$f'(x) = 4(x + 167)(x + 2)(x - 169),$$
$$f''(x) = 12(x - 97)(x + 97),$$
$$f'''(x) = 24x.$$

The most well known, open problem associated with rational derived polynomials is determining whether or not there exists a rational derived quartic polynomial with distinct roots. The reader should consult Buchholz and Kelly [2], Buchholz and MacDougall [3], and Stroeker [9] for background on this problem. We may also consider polynomials with rational roots, whose first $l$ derivatives also have rational roots. The notation $\mathcal{D}(n, l, \mathbb{Q})$, introduced in [3], denotes the set of polynomials of degree $n$ such that they and their first $l$ derivatives have rational roots. Quartic polynomials with four roots in an integral domain $\mathcal{D}$ and whose first
derivatives have three roots in \( \mathbb{D} \) were treated by Groves in [6] and in [7]. In this paper we study polynomials of the type

\[ f(x) = f_{n,a}(x) = x^n(x-1)(x-a) \]

where the integer \( n \) satisfies \( n \geq 3 \) and \( a \neq 0, 1 \) is a rational number. Our main result confirms that for each integer \( n \geq 3 \) there exist infinitely many rational numbers \( a \neq 0, 1 \) such that \( f_{n,a}(x) \in \overline{D}(n+2,2,\mathbb{Q}) \). We state this in Theorem 1 and give a proof in Section 2. In Section 3 we consider the sets \( \overline{D}(n+2,l,\mathbb{Q}) \) where \( l \geq 3 \), finding two pairs \((n,a)\) such that \( f_{n,a}(x) \in \overline{D}(n+2,3,\mathbb{Q}) \), then finish by asking two questions.

We now state our main theorem. The case where \( n = 2 \) of our theorem is covered in [2] and [3].

**Theorem 1.** For each integer \( n \geq 3 \), there exist infinitely many rational numbers \( a \) such that \( f_{n,a}(x) \in \overline{D}(n+2,2,\mathbb{Q}) \). For fixed \( n \), these values of \( a \) have the form

\[
a = \frac{(2n^2 + kn + 4n + k)(kn + k - 2)}{2(n+1)^3k},
\]

where \( k \) satisfies

\[
c_4k^4 + c_3k^3 + c_2k^2 + c_1k + c_0 = Y^2 \tag{2}
\]

for some rational number \( Y \), with

\[
\begin{align*}
c_0 &= 16n^4(n+1)^2(n+2)^2, \\
c_1 &= -32n^2(n+1)^3(n+2)^2, \\
c_2 &= 8n(n+1)^4(n+2)(3n^2 + 4n - 2), \\
c_3 &= 8n(n+1)^3(n+2), \\
c_4 &= n^2(n+1)^6.
\end{align*}
\]

If we set

\[
a^* = \frac{(n^3 + 2n^2 - 2)(n^3 + 2n^2 + 2n + 2)}{2n(n+1)^2(n^3 + 2n^2 - 2n - 2)},
\]

then for each \( n \geq 3 \), \( f_{n,a^*} \in \overline{D}(n+2,2,\mathbb{Q}) \).

**2. Proof Of Theorem**

We will derive, for each \( n \geq 3 \), a parametrizing elliptic curve that specifies the values of \( a \) described in our theorem. A confirmation that these curves have positive rank will be given as well. Finally, by finding a point \((x(n), y(n))\) on this family of curves, we will obtain the value of \( a^* \) given in our theorem.
Proof. The first derivative of \( f(x) \) is given by
\[
f'(x) = x^{n-1}((n+2)x^2 - (n+1)(a+1)x + na).
\]
In order for \( f'(x) \) to have rational roots, the discriminant of the quadratic factor of \( f'(x) \) must equal the square of a rational number. This discriminant, written as a quadratic in \( a \), is given by
\[
(n+1)^2a^2 - (2n^2 + 4n - 2)a + (n+1)^2.
\]
Using a standard approach, we parametrize the values of \( a \) for which (4) is equal to a rational square and obtain
\[
a = \frac{(2n^2 + kn + 4n + k)(kn + k - 2)}{2(n+1)^3k},
\]
where \( k \) denotes a nonzero rational number. This establishes (1). Assuming that \( a \) has the value given by (1), substituting it into \( f(x) \), then examining the second derivative \( f''(x) \), we find another quadratic factor whose discriminant must equal the square of a rational number. This discriminant, written as a quartic in \( f \), is
\[
n^2(n+1)^6k^4 + 8n(n+1)^5(n+2)k^3 + 8n(n+1)^4(n+2)(3n^2 + 4n - 2)k^2
\]
\[
-32n^2(n+1)^3(n+2)^2k + 16n^4(n+1)^2(n+2)^2.
\]
Requiring this discriminant (5) to equal \( Y^2 \) for some rational number \( Y \) establishes (2). Converting (2) to Weierstrass form yields
\[
y^2 = x^3 - 27n^2(n+1)^2(n+2)^2(3n^2 + 1)x + 54n^3(n+1)^3(n+2)^3(3n-1)(3n+1).
\]
For \( n \geq 3 \), the elliptic curve (6) has positive rank. To see this, note that for fixed \( n \geq 3 \), the cubic on the right hand side of (6) has three rational roots contributing three points of order two in the group \( \Gamma \) of rational points of (6). Therefore, from the known list of torsion subgroups [8] we conclude that the torsion subgroup of \( \Gamma \) is of the form
\[
\mathbb{Z}_2 \times \mathbb{Z}_{2N}, \quad 1 \leq N \leq 4.
\]
Consider the point
\[
P = (-3(n+2)(3n^2 + n - 6)(n+1), 54n(n-1)(n+1)^2(n+2)^2)
\]
in \( \Gamma \). \( P \) has infinite order, as can be deduced by noting (7), forming the set \( \{P, 2P, 4P\} \), and checking that this set contains neither points of order 2, nor a point and its inverse. Thus the elliptic curve (6) has positive rank for \( n \geq 3 \), implying that there exists infinitely many rational values of \( a \) such that \( f_{n,a}(x) \in \overline{D(n+2,2,\mathbb{Q})} \). Converting the point \( P \) by means of birational transformations to a point on the quartic curve (2) yields a \( k \)-value of
\[
k = \frac{4n^2(n+2)}{(n+1)(n^3 + 2n^2 - 2n - 2)}.
\]
Substituting for \( k \) from (8) into (1) yields the value of \( a^* \) given in the statement of our theorem. The fact that \( a^* \) has the property stated in our theorem is a consequence of the following calculation.

\[
f(x) = x^n(x - 1) \left( x - \frac{(n^3 + 2n^2 - 2)(n^3 + 2n^2 + 2n + 2)}{2n(n + 1)^2(n^3 + 2n^2 - 2n - 2)} \right),
\]

\[
f'(x) = (n + 2)x^{n-1} \left( x - \frac{n^3 + 2n^2 + 2n + 2}{2n(n + 2)(n + 1)} \right) \left( x - \frac{n(n^3 + 2n^2 - 2)}{(n + 1)(n^3 + 2n^2 - 2n - 2)} \right),
\]

\[
f''(x) = (n + 1)(n + 2)x^{n-2} \left( x - \frac{n^3 + 2n^2 - 2}{2(n + 2)(n + 1)^2} \right) \left( x - \frac{(n - 1)(n^3 + 2n^2 + 2n + 2)}{(n + 1)(n^3 + 2n^2 - 2n - 2)} \right).
\]

### 3. Rational Roots of Higher Derivatives

It was shown by Flynn [5] that if \( n = 3 \), then \( f_{n,a}(x) \notin \mathcal{D}(n + 2, 3, \mathbb{Q}) \) for all rational values of \( a \neq 0, 1 \). The idea of the proof was to form the product of the discriminants of the irreducible quadratic factors of the first three derivatives of \( f_{3,a}(x) \), and require this product to be a square in \( \mathbb{Q} \). This leads to a study of the rational points on a genus 2 curve, from which the result for \( n = 3 \) was deduced. We formed the corresponding products of discriminants for \( 4 \leq n \leq 1500 \) and searched for rational points on the resulting genus 2 curves using the routine "RationalPoints" in Magma [1]. For these values of \( n \), no rational numbers \( a \neq 0, 1 \) were found for which \( f_{n,a}(x) \in \mathcal{D}(n + 2, 3, \mathbb{Q}) \). As an alternative approach, we assumed that for some \( n \geq 4 \), and some rational number \( a \neq 0, 1 \), the third derivative of \( f_{n,a}(x) \) has the rational root \( x = 1 \). Substituting \( x = 1 \) into \( f'''(x) \) produces an equation for \( a \) which yields

\[
a = \frac{n + 1}{n - 1}.
\]

By substituting this value of \( a \) into \( f_{n,a}(x) \), we find that first two derivatives of \( f_{n,a}(x) \) each have a quadratic factor, which for purposes of having rational roots, must have their discriminants equal to the square of a rational number. These discriminants are, respectively,

\[
8n(n + 1)
\]

and

\[
4n(3n - 2).
\]

Since \( \gcd(n, n + 1) = 1 \), forcing (9) to equal a square in \( \mathbb{Q} \) implies that

\[
(n, n + 1) = (2a^2, b^2) \text{ or } (a^2, 2b^2)
\]

for positive integers \( a \) and \( b \). Combining (10) and (11) we deduce that either

\[
4(b^2 - 1)(3b^2 - 5)
\]
or
\[ 4(2b^2 - 1)(6b^2 - 5) \]
is equal to a square in \( \mathbb{Q} \). The positive integral values of \( b \) for which (12) or (13) is equal to a square in \( \mathbb{Q} \) can be determined by using the command "IntegralQuarticPoints" available in Magma [1]. We find that for \( b = 29 \), (13) yields a square from which we determine \( n = 1681 \). This leads to the following example of a degree 1683 polynomial \( f_{n,a}(x) \in \overline{D}(1683,3,\mathbb{Q}) \).

\[
\begin{align*}
f(x) &= x^{1681}(x-1) \left( x - \frac{841}{840} \right), \\
f'(x) &= \frac{x^{1680}(1190x - 1189)(1188x - 1189)}{840}, \\
f''(x) &= \frac{841x^{1679}(616x - 615)(2295x - 2296)}{420}, \\
f'''(x) &= 2827442x^{1678}(x-1)(1683x - 1679).
\end{align*}
\]

We also substituted \( n = 1681 \) into the product of discriminants mentioned at the beginning of this section. Using Magma, we searched for values of \( a \) for which this product of discriminants was equal to a rational square. We found two values of \( a \), namely \( a = \frac{841}{840} \) and \( a = \frac{840}{841} \). The first value of \( a \) gave rise to the previous example. The second value of \( a \) leads to another polynomial in \( \overline{D}(1683,3,\mathbb{Q}) \) that is a scaled version of the previous example.

\[
\begin{align*}
\begin{align*}
f(x) &= x^{1681}(x-1) \left( x - \frac{840}{841} \right), \\
f'(x) &= \frac{x^{1680}(493x - 492)(2871x - 2870)}{841}, \\
f''(x) &= \frac{2x^{1679}(9251x - 9225)(128673x - 128576)}{841}, \\
f'''(x) &= \frac{10086x^{1678}(841x - 840)(471801x - 470120)}{841}.
\end{align*}
\end{align*}
\]

We finish with two questions. Are there infinitely many pairs \((n, a)\), where \( n \geq 3 \) is an integer and \( a \neq 0, 1 \) is a rational number such that \( f_{n,a}(x) \in \overline{D}(n + 2,3,\mathbb{Q}) \)? Does there exist an integer \( n \geq 3 \) and a rational number \( a \neq 0, 1 \) such that \( f_{n,a}(x) \in \overline{D}(n + 2,4,\mathbb{Q}) \)?

**References**


