Certain Classes of Harmonic Functions Associated with Dual Convolution

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Abstract. In this paper, we investigate several properties for the harmonic classes \( \mathcal{M}_H(\varphi, \psi; t, \beta, \sigma) \) and \( \mathcal{M}_H(\varphi, \psi; t, \beta) \). We obtain coefficient bounds, distortion theorem, extreme points, convolution condition, convex combinations and integral operator for these classes.

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1. Introduction

A continuous complex valued functions \( f = u + iv \) which is defined in a simply connected complex domain \( \mathcal{D} \) is said to be harmonic in \( \mathcal{D} \) if both \( u \) and \( v \) are real harmonic in \( \mathcal{D} \). In any simply connected domain we can write

\[
f(z) = h(z) + g(z),
\]

(1)

where \( h \) and \( g \) are analytic in \( \mathcal{D} \). We call \( h \) the analytic part and \( g \) the co-analytic part of \( f \). A necessary and sufficient condition for \( f \) to be locally univalent and sense-preserving in \( \mathcal{D} \) is that \( |h'(z)| > |g'(z)| \) in \( \mathcal{D} \) (see [7]).

Let \( \mathcal{A} \) denote the class of the functions \( f \) of the form:

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k,
\]

which are analytic in the open unit disc \( U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \} \) and satisfy the normalization condition \( f(0) = f'(0) - 1 = 0 \).

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For $1 < \beta \leq \frac{4}{3}$ and $z \in U$, let
\[
\mathcal{M}(\beta) = \left\{ f \in A : \Re \left( \frac{zf'(z)}{f(z)} \right) < \beta \right\},
\]
and
\[
\mathcal{N}(\beta) = \left\{ f \in A : \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) < \beta \right\}.
\]
These classes $\mathcal{M}(\beta), \mathcal{N}(\beta)$ were extensively studied by Uralegaddi et al. [19], see also Owa and Srivastava [13], Porwal and Dixit [16] and Breaz [6].

Denote by $\mathcal{S}_H$, the class of functions $f$ of the form (1) that are harmonic univalent and sense preserving in the unit disc $U = \{ z : |z| < 1 \}$ for which $f(0) = f_z(0) - 1 = 0$. For $f = h + \overline{g} \in \mathcal{S}_H$, we may express
\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} b_k z^k, |b_1| < 1, \tag{2}
\]
where the analytic functions $h$ and $g$ are of the form:
\[
h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, |b_1| < 1. \tag{3}
\]

In 1984 Clunie and Sheil-Small [7] investigated the class $\mathcal{S}_H$ as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on $\mathcal{S}_H$ and its subclasses. For more basic results one may refer to the following standard introductory, Porwal [15, Chapter 5] defined the subclass $\mathcal{M}_H(\beta) \subset \mathcal{S}_H$ consisting of harmonic univalent functions $f(z)$ satisfying the following condition:
\[
\mathcal{M}_H(\beta) = \left\{ f(z) \in \mathcal{S}_H : \Re \left( \frac{zh'(z) - zg'(z)}{h(z) + g(z)} \right) < \beta \right\} \left( 1 < \beta \leq \frac{4}{3}; z \in U \right),
\]
He proved that if $f = h + \overline{g}$, where $h$ and $g$ are given by (3) and if
\[
\sum_{k=2}^{\infty} \frac{(k-\beta)}{\beta - 1} |a_k| + \sum_{k=1}^{\infty} \frac{(k+\beta)}{\beta - 1} |b_k| \leq 1 \left( 1 < \beta \leq \frac{4}{3} \right), \tag{4}
\]
then $f(z) \in \mathcal{M}_H(\beta)$.

For $g \equiv 0$ the class of $\mathcal{M}_H(\beta)$ is reduced to the class $\mathcal{M}(\beta)$ studied by Uralegaddi et al. [19].

The convolution of two functions of the form
\[
\varphi(z) = z + \sum_{k=2}^{\infty} \lambda_k z^k \quad (\lambda_k \geq 0) \quad \text{and} \quad \psi(z) = z + \sum_{k=2}^{\infty} \mu_k z^k \quad (\mu_k \geq 0), \tag{5}
\]
is defined as
\[(\varphi \ast \psi)(z) = z + \sum_{k=2}^{\infty} \lambda_k M_k z^k = (\psi \ast \varphi)(z),\] (6)
while the integral convolution is defined by
\[(\varphi \check{\ast} \psi)(z) = z + \sum_{k=2}^{\infty} \frac{\lambda_k M_k}{k} z^k = (\psi \check{\ast} \varphi)(z).\] (7)

Motivated by the work of Ahuja [1], we consider the class \(\mathcal{M}_{\mathcal{H}}(\varphi, \psi; t, \beta, \sigma)\) of functions of the form (1) satisfying the condition
\[
\text{Re} \left\{ \frac{h(z) \ast \varphi(z) - \sigma g(z) \ast \psi(z)}{(1 - t)z + t [h(z) \check{\ast} \varphi(z) + \sigma g(z) \check{\ast} \psi(z)]} \right\} < \beta,
\] (8)
where \(0 \leq t \leq 1, |\sigma| = 1, 1 < \beta \leq \frac{4}{3}\), \(\varphi(z)\) and \(\psi(z)\) are given by (5).

We note that:

i) \(\mathcal{M}_{\mathcal{H}}\left(\frac{z}{(1 - z)^2}, \frac{z}{(1 - z)^2}; 1, \beta, 1\right) = \mathcal{M}_{\mathcal{H}}(\beta)\) (see [15]);

ii) \(\mathcal{M}_{\mathcal{H}}(z + \sum_{k=2}^{\infty} k \Gamma_k (\alpha_1) z^k, z + \sum_{k=2}^{\infty} k \Gamma_k (\alpha_1) z^k; 1, \beta, 1) = \mathcal{M}_{\mathcal{H}}(\alpha_1, \beta),\)

\((\alpha_i > 0, i = 1, \ldots, q; \beta_j > 0, j = 1, \ldots, s; q \leq s + 1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, \ldots\})\)

where (see [14])
\[
\Gamma_k (\alpha_1) = \frac{(\alpha_1)_{k-1}(\alpha_2)_{k-1} \cdots (\alpha_q)_{k-1}}{(\beta_1)_{k-1}(\beta_2)_{k-1} \cdots (\beta_s)_{k-1}} \cdot \frac{1}{(k-1)!} (k \geq 2).
\]

Also we note that:

i) \(\mathcal{M}_{\mathcal{H}}\left(\frac{z + z^2}{(1 - z)^3}, \frac{z + z^2}{(1 - z)^3}; 1, \beta, -1\right) = \mathcal{M}_{\mathcal{H}}(\beta) = \text{Re} \left\{ \frac{z^2 h''(z) + z h'(z) + z^2 g''(z) + z g'(z)}{zh'(z) - zg'(z)} \right\} < \beta;\)

ii) \(\mathcal{M}_{\mathcal{H}}(z + \sum_{k=2}^{\infty} k^{n+1} z^k, z + \sum_{k=2}^{\infty} k^{n+1} z^k; 1, \beta, (-1)^n)\)
for $n \in \mathbb{N}_0$ and where $D^n$ is the modified Salagean differential operator (see [11, 18, 20]);

$$\mathcal{M}_{\mathfrak{M}}(\beta, n) = \text{Re} \left\{ \frac{D^{n+1}h(z) + (-1)^{n+1}D^{n+1}g(z)}{D^n h(z) + (-1)^n D^n g(z)} \right\} < \beta,$$

for $n \in \mathbb{N}_0$ and where $I^n$ is the modified Salagean integral operator (see [8], with $p = 1$, also see [18]);

$$\mathcal{M}_{\mathfrak{M}}(z + \sum_{k=2}^{\infty} k^{-n}z^k, z + \sum_{k=2}^{\infty} k^{-n}z^k; 1, \beta, (-1)^{n+1})$$

$$= \mathcal{L}_{\mathfrak{M}}(\beta, n) = \text{Re} \left\{ \frac{I^{n+1}h(z) + (-1)^n I^{n+1}g(z)}{I^{n+1} h(z) + (-1)^n I^{n+1} g(z)} \right\} < \beta,$$

for $n \in \mathbb{N}_0$ and where $I^n$ is the modified Salagean integral operator (see [2], with $p = 1$);

$$\mathcal{M}_{\mathfrak{M}}(z + \sum_{k=2}^{\infty} k \left[ 1 + \lambda (k - 1) \right]^{-n} z^k, z + \sum_{k=2}^{\infty} k \left[ 1 + \lambda (k - 1) \right]^{-n} z^k; 1, \beta, (-1)^n)$$

$$= \mathcal{L}_{\mathfrak{M}}(\beta, n, \lambda) = \text{Re} \left\{ \frac{z \left( \frac{D^n h(z)}{I^n h(z)} \right)' - (-1)^n z \left( \frac{D^n g(z)}{I^n g(z)} \right)'}{D^n h(z) + (-1)^n D^n g(z)} \right\} < \beta,$$

where $\lambda \geq 0$, $n \in \mathbb{N}_0$, and $D^n_{\alpha}$ is the modified Al-Oboudi operator (see [2, 21], also see [3], with $p = 1$);

$$\mathcal{M}_{\mathfrak{M}}(z + \sum_{k=2}^{\infty} k \left[ 1 + \lambda (k - 1) \right]^{-n} z^k, z + \sum_{k=2}^{\infty} k \left[ 1 + \lambda (k - 1) \right]^{-n} z^k; 1, \beta, (-1)^n)$$

$$= \mathcal{L}_{\mathfrak{M}}(\beta, n, \lambda) = \text{Re} \left\{ \frac{z \left( \frac{I^n h(z)}{I^n h(z)} \right)' - (-1)^n z \left( \frac{I^n g(z)}{I^n g(z)} \right)'}{I^n h(z) + (-1)^n I^n g(z)} \right\} < \beta,$$

for $\lambda \geq 0$ and $n \in \mathbb{N}_0$, and where $I^n_{\alpha}$ is modified integral operator see ([4], with $p = 1$, also see [10], with $\ell = 0$);

$$\mathcal{M}_{\mathfrak{M}} \left( z + \sum_{k=2}^{\infty} k \left( \frac{1 + \ell + \lambda(k-1)}{1+\ell} \right)^m z^k, z + \sum_{k=2}^{\infty} k \left( \frac{1 + \ell + \lambda(k-1)}{1+\ell} \right)^m z^k; 1, \beta, (-1)^n \right)$$
Theorem 1. Let \( f \in M_{\mathcal{K}}(\varphi, \psi; t, \beta, \sigma) \) be functions of the form:

\[
M_{\mathcal{K}}(\varphi, \psi; t, \beta, \sigma) \subset H = \{0, \pm 1, \ldots \}, \text{ and } J^m(\lambda, \ell) \text{ is the modified Prajapat operator (see [9, 17], with } p = 1).
\]

Further, let for \( \sigma = 1, M_{\mathcal{K}}(\varphi, \psi; t, \beta, \sigma) \) be the subclass of \( M_{\mathcal{K}}(\varphi, \psi; t, \beta, \sigma) \) consisting of functions of the form:

\[
f(z) = z + \sum_{k=1}^{\infty} a_k z^k - \sum_{k=1}^{\infty} b_k \lambda^k (|b_1| < 1).
\]

In this paper, we obtained the coefficient bounds for the classes \( M_{\mathcal{K}}(\varphi, \psi; t, \beta, \sigma) \) and \( M_{\mathcal{K}}(\varphi, \psi; t, \beta) \). We also obtain distortion theorem, extreme points, convolution, convex combinations and integral operator for functions in the class \( M_{\mathcal{K}}(\varphi, \psi; t, \beta) \).

2. Coefficient Bounds and Distortion Theorem

Unless otherwise mentioned, we assume in the reminder of this paper that \( 0 \leq t \leq 1, |\sigma| = 1, 1 < \beta \leq \frac{4}{3} \) and \( z \in U \). We begin with a sufficient condition for functions in the class \( M_{\mathcal{K}}(\varphi, \psi; t, \beta, \sigma) \) and obtain distortion theorem for functions in the class \( M_{\mathcal{K}}(\varphi, \psi; t, \beta) \).

Theorem 1. Let \( f = h + \frac{g}{h} \), where \( h \) and \( g \) are given by (3), and satisfy the condition

\[
\sum_{k=2}^{\infty} \lambda_k (k - t \beta) a_k + \sum_{k=1}^{\infty} \mu_k (k + t \beta) b_k \leq \beta - 1,
\]

where

\[
k^2 (\beta - 1) \leq \lambda_k (k - t \beta) \text{ and } k^2 (\beta - 1) \leq \mu_k (k + t \beta) \text{ for } k \geq 2.
\]

Then \( f(z) \) is sense-preserving, harmonic univalent in \( U \) and \( f(z) \in M_{\mathcal{K}}(\varphi, \psi; t, \beta, \sigma) \).

Proof. If \( z_1 \neq z_2 \), then by using (11), we have

\[
\left| \frac{f(z_2) - f(z_1)}{h(z_2) - h(z_1)} \right| \geq 1 - \left| \frac{g(z_2) - g(z_1)}{h(z_2) - h(z_1)} \right| \geq 1 - \left| \frac{\sum_{k=1}^{\infty} b_k \left( z_2^k - z_1^k \right)}{z_2 - z_1} \right| \geq 0,
\]

where

\[
1 - \sum_{k=2}^{\infty} k |a_k| \geq 1 - \sum_{k=1}^{\infty} \lambda_k \left( \frac{z_2 - z_1}{\beta - 1} \right) b_k \geq 0.
\]
which proves the univalent. Also $f$ is sense-preserving in $U$ since
\[
|h'(z)| \geq 1 - \sum_{k=2}^{\infty} k |a_k| |z|^{k-1} > 1 - \sum_{k=2}^{\infty} k |a_k|
\]
\[
\geq 1 - \sum_{k=2}^{\infty} \frac{\lambda_k}{k} \left( \frac{k - t\beta}{\beta - 1} \right) |a_k|
\]
\[
\geq \sum_{k=1}^{\infty} \frac{\mu_k}{k} \left( \frac{k + t\beta}{\beta - 1} \right) |b_k|
\]
\[
\geq \sum_{k=1}^{\infty} k |b_k| |z|^k \geq |g'(z)|.
\]

Now we show that $f \in \mathcal{M}_{\varphi, \psi; t, \beta, \sigma}$. We only need to show that if (10) holds then the condition (8) is satisfied, then we want to prove that
\[
\left| \frac{h(z) + \varphi(z) - \sigma g(z) + \psi(z)}{(1-t)z + t} \frac{h(z) + \varphi(z) + \sigma g(z) + \psi(z)}{(1-t)z + t} \right| - 1 < 1, \ z \in U.
\]
We have
\[
\left| \frac{h(z) + \varphi(z) - \sigma g(z) + \psi(z)}{(1-t)z + t} \frac{h(z) + \varphi(z) + \sigma g(z) + \psi(z)}{(1-t)z + t} \right| - (2\beta - 1)
\]
\[
\leq \frac{\sum_{k=2}^{\infty} \frac{\lambda_k}{k} (k - t) |a_k| + \sum_{k=1}^{\infty} \frac{\mu_k}{k} (k + t) |b_k|}{2(\beta - 1) - \sum_{k=2}^{\infty} \frac{\lambda_k}{k} (k - 2\beta t + t) |a_k| - \sum_{k=1}^{\infty} \frac{\mu_k}{k} (k + 2\beta t - t) |b_k|}.
\]
The last expression is bounded above by 1, if
\[
\sum_{k=2}^{\infty} \frac{\lambda_k}{k} (k - t) |a_k| + \sum_{k=1}^{\infty} \frac{\mu_k}{k} (k + t) |b_k|
\]
\[
\leq 2(\beta - 1) - \sum_{k=2}^{\infty} \frac{\lambda_k}{k} (k - 2\beta t + t) |a_k| - \sum_{k=1}^{\infty} \frac{\mu_k}{k} (k + 2\beta t - t) |b_k|,
\]
which is equivalent to
\[
\sum_{k=1}^{\infty} \frac{\lambda_k}{k} (k - t\beta) |a_k| + \sum_{k=1}^{\infty} \frac{\mu_k}{k} (k + t\beta) |b_k| \leq \beta - 1.
\] (12)
But (12) is true by hypothesis and the Theorem is proved. \hfill \Box

In the following theorem, it is shown that the condition (10) is also necessary for function \( f(z) \) given by (9) and belongs to \( M\varphi (\varphi , \psi ; t, \beta ) \).

**Theorem 2.** Let the function \( f(z) \) given by (9). Then \( f(z) \in M\varphi (\varphi , \psi ; t, \beta ) \), if and only if the coefficient bound (10) holds.

*Proof.* Since \( M\varphi (\varphi , \psi ; t, \beta , \sigma ) \subseteq M\varphi (\varphi , \psi ; t, \beta ) \), we only need to prove the only if part of the theorem. To this end for functions \( f \in M\varphi (\varphi , \psi ; t, \beta ) \), we notice that the necessary and sufficient condition to be in the class \( M\varphi (\varphi , \psi ; t, \beta ) \) is that

\[
\text{Re} \left\{ \frac{h(z) \ast \varphi(z) - g(z) \ast \psi(z)}{(1-t)z + t \left[ h(z)\hat{\varphi}(z) + g(z)\hat{\psi}(z) \right]} \right\} < \beta.
\]

This is equivalent to

\[
\text{Re} \left\{ \frac{(\beta - 1)z - \sum_{k=2}^{\infty} \frac{\lambda_k}{k} (k - t \beta) \left| a_k \right| z^k - \sum_{k=1}^{\infty} \frac{\mu_k}{k} (k + t \beta) \left| b_k \right| z^{k-1}}{z + \sum_{k=2}^{\infty} \frac{\lambda_k}{k} \left| a_k \right| z^k - \sum_{k=1}^{\infty} \frac{\mu_k}{k} \left| b_k \right| z^{k-1}} \right\} \geq 0.
\]

The above condition must hold for all values of \( z \in U \), so that on taking \( z = r < 1 \), the above inequality reduces to

\[
\frac{(\beta - 1) - \sum_{k=2}^{\infty} \frac{\lambda_k}{k} (k - t \beta) \left| a_k \right| r^{k-1} - \sum_{k=1}^{\infty} \frac{\mu_k}{k} (k + t \beta) \left| b_k \right| r^{k-1}}{1 + \sum_{k=2}^{\infty} \frac{\lambda_k}{k} \left| a_k \right| r^{k-1} - \sum_{k=1}^{\infty} \frac{\mu_k}{k} \left| b_k \right| r^{k-1}} \geq 0.
\]

If the condition (10) does not hold then the numerator of (14) is negative for \( r \) and sufficiently close to 1. Thus there exists a \( z_0 = r_0 \) in \((0,1)\) for which the quotient in (14) is negative. This contradicts the required condition for \( f \in M\varphi (\varphi , \psi ; t, \beta ) \). This completes the proof of Theorem. \hfill \Box

**Theorem 3.** Let the function \( f(z) \) given by (9) be in the class \( M\varphi (\varphi , \psi ; t, \beta ) \) and

\[
A_k \leq \frac{\lambda_k}{k} (k - t \beta), \quad B_k \leq \frac{\mu_k}{k} (k + t \beta)
\]

for \( k \geq 2 \), \( C = \min \{A_2, B_2\} \). Then for \( |z| = r < 1 \), we have

\[
|f(z)| \leq (1 + |b_1|)r + \left( \frac{\beta - 1}{C} - \frac{\beta + 1}{C} |b_1| \right) r^2,
\]

and

\[
|f(z)| \geq (1 - |b_1|)r - \left( \frac{\beta - 1}{C} - \frac{\beta + 1}{C} |b_1| \right) r^2.
\]

The equalities in (15) and (16) are attained for the functions \( f \) given by

\[
f(z) = (1 + |b_1|)z + \left( \frac{\beta - 1}{C} - \frac{\beta + 1}{C} |b_1| \right) z^2.
\]
and

$$f(z) = (1 - |b_1|)z - \left(\frac{\beta - 1}{C} - \frac{\beta + 1}{C} |b_1|\right)z^2$$  \hspace{1cm} (18)

where $|b_1| \leq \frac{\beta - 1}{\beta + 1}$.

**Proof.** Let $f(z) \in \overline{M}_{\varphi, \psi}(\varphi, \psi; t, \beta)$, then we have

$$|f(z)| \leq (1 + |b_1|)r + \sum_{k=2}^{\infty}(|a_k| + |b_k|)r^k$$

$$\leq (1 + |b_1|)r + \sum_{k=2}^{\infty}(|a_k| + |b_k|)r^2$$

$$= (1 + |b_1|)r + \frac{\beta - 1}{C} \sum_{k=2}^{\infty} \left(\frac{C}{\beta - 1} |a_k| + \frac{C}{\beta - 1} |b_k|\right)r^2$$

$$\leq (1 + |b_1|)r + \frac{\beta - 1}{C} \left(1 - \frac{(\beta + 1)|b_1|}{\beta - 1}\right)r^2$$

$$= (1 + |b_1|)r + \left(\frac{\beta - 1}{C} - \frac{(\beta + 1)|b_1|}{C}\right)r^2,$$

which proves the assertion (15) of Theorem 3. The proof of the assertion (16) is similar, thus, we omit it. \qed

**Remark 1.** Putting $\varphi = \psi = z + \sum_{k=2}^{\infty} k \Gamma_k(a_1)z^k$, $\lambda_2 = \mu_2 = 2 \Gamma_2(a_1)$, $t = 1$ and $C = 2 - \beta$ in Theorem 3, we improve the result obtained by Pathak et al. [14, Theorem 2.4], by adding the condition $|b_1| \leq \frac{\beta - 1}{\beta + 1}$.

The following covering result follows the left hand inequality Theorem 3.

**Corollary 1.** Let the function $f(z)$ given by (9) be in the class $\overline{M}_{\varphi, \psi}(\varphi, \psi; t, \beta)$, where $|b_1| < \frac{C - (\beta - 1)}{C - (\beta + 1)}$ and $A_k \leq \frac{\lambda_k}{k} (k - t \beta)$, $B_k \leq \frac{\mu_k}{k} (k + t \beta)$ for $k \geq 2$, $C = \min \{A_2, B_2\}$. Then for $|z| = r < 1$, we have

$$\left\{w : |w| < \frac{C - (\beta - 1)}{C} - \frac{C - (\beta + 1)}{C} |b_1|\right\} \subset f(U).$$

3. Extreme Points

In this section we determine the extreme points of the closed convex hull of the class $\overline{M}_{\varphi, \psi}(\varphi, \psi; t, \beta)$ denoted by $\text{clco} \overline{M}_{\varphi, \psi}(\varphi, \psi; t, \beta)$. 

Theorem 4. Let \( f(z) \) given by (9), Then \( f(z) \in \text{clco} \ \mathcal{M}_{\varphi, \psi}(\varphi, \psi; t, \beta) \) if and only if

\[
f(z) = \sum_{k=1}^{\infty} \left[ X_k h_k(z) + Y_k g_k(z) \right],
\]

where

\[
h_1(z) = z,
\]

\[
h_k(z) = z + \frac{(\beta - 1)k}{\lambda_k(k-t\beta)} z^k \quad (k \geq 2),
\]

and

\[
g_k(z) = z - \frac{(\beta - 1)k}{\mu_k(k+t\beta)} \bar{z}^k \quad (k \geq 1),
\]

where \( \sum_{k=1}^{\infty} (X_k + Y_k) = 1, X_k \geq 0 \) and \( Y_k \geq 0 \). In particular, the extreme points of the class \( \mathcal{M}_{\varphi, \psi}(\varphi, \psi; t, \beta) \) are \( \{h_k\} \) \( (k \geq 2) \) and \( \{g_k\} \) \( (k \geq 1) \), respectively.

Proof. For a function \( f(z) \) of the form (19), we have

\[
f(z) = \sum_{k=1}^{\infty} \left[ X_k h_k(z) + Y_k g_k(z) \right]
\]

\[
= \sum_{k=1}^{\infty} X_k \left( z + \frac{(\beta - 1)k}{\lambda_k(k-t\beta)} z^k \right) + Y_k \left( z - \frac{(\beta - 1)k}{\mu_k(k+t\beta)} \bar{z}^k \right)
\]

\[
= z + \sum_{k=2}^{\infty} \frac{(\beta - 1)k}{\lambda_k(k-t\beta)} X_k z^k - \sum_{k=1}^{\infty} \frac{(\beta - 1)k}{\mu_k(k+t\beta)} Y_k \bar{z}^k.
\]

But,

\[
\sum_{k=2}^{\infty} \left( \frac{\lambda_k(k-t\beta)}{k(1-\beta)} \cdot \frac{k(\beta - 1)}{\lambda_k(k-t\beta)} X_k \right)
\]

\[
+ \sum_{k=1}^{\infty} \left( \frac{\mu_k(k+t\beta)}{k(1+\beta)} \cdot \frac{k(\beta - 1)}{\mu_k(k+t\beta)} Y_k \right)
\]

\[
= \sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k = 1 - X_1 \leq 1.
\]

Thus \( f(z) \in \text{clco} \ \mathcal{M}_{\varphi, \psi}(\varphi, \psi; t, \beta) \).

Conversely, assume that \( f(z) \in \text{clco} \ \mathcal{M}_{\varphi, \psi}(\varphi, \psi; t, \beta) \). Set

\[
X_k = \frac{\lambda_k(k-t\beta)}{k(\beta - 1)} \ |a_k| \quad (0 \leq X_k \leq 1; k \geq 2),
\]

where \( \sum_{k=1}^{\infty} (X_k + Y_k) = 1, X_k \geq 0 \) and \( Y_k \geq 0 \).
\[ Y_k = \frac{\mu_k (k + t\beta)}{k(\beta - 1)} \left| b_k \right| \quad (0 \leq Y_k \leq 1; k \geq 1) \]

and \( X_1 = 1 - \sum_{k=2}^{\infty} X_k - \sum_{k=1}^{\infty} Y_k \). Therefore,

\[
\begin{align*}
Y_k & = \mu_k (k + t\beta) \\
& \quad \left( k(\beta - 1) \right) \\
\end{align*}
\]

\[
b_k \quad (0 \leq Y_k \leq 1; k \geq 1)
\]

This completes the proof of Theorem. \( \square \)

4. Convolution and Convex Combination

In this section, we determine the convolution properties and convex combination. Let the functions \( f_m(z) \) define by

\[
f_m(z) = z + \sum_{k=2}^{\infty} a_{k,m} z^k - \sum_{k=1}^{\infty} b_{k,m} z^k \quad (m = 1, 2), \tag{23}
\]

are in the class \( \mathcal{M}_{\mathcal{H}}(\varphi, \psi; t, \beta) \), we denote by \((f_1 \ast f_2)(z)\) the convolution or (Hadamard Product) of the function \( f_1(z) \) and \( f_2(z) \), that is,

\[
(f_1 \ast f_2)(z) = z + \sum_{k=2}^{\infty} a_{k,1} a_{k,2} z^k - \sum_{k=1}^{\infty} b_{k,1} b_{k,2} z^k, \tag{24}
\]

while the integral convolution is defined by

\[
(f_1 \diamond f_2)(z) = z + \sum_{k=2}^{\infty} \frac{a_{k,1}}{k} a_{k,2} z^k - \sum_{k=1}^{\infty} \frac{b_{k,1}}{k} b_{k,2} z^k. \tag{25}
\]

We first show that the class \( \mathcal{M}_{\mathcal{H}}(\varphi, \psi; t, \beta) \) is closed under convolution.
Theorem 5. For \(1 < \beta \leq \delta \leq \frac{4}{3}\), let the functions \(f_1 \in \overline{H}(\varphi, \psi; t, \beta)\) and \(f_2 \in \overline{H}(\varphi, \psi; t, \delta)\). Then

\[
(f_1 \ast f_2)(z) \in \overline{H}(\varphi, \psi; t, \beta) \subset \overline{H}(\varphi, \psi; t, \delta),
\]

(26)

\[
(f_1 \circ f_2)(z) \in \overline{H}(\varphi, \psi; t, \beta) \subset \overline{H}(\varphi, \psi; t, \delta).
\]

(27)

Proof. Let \(f_m(z)(m = 1, 2)\) are given by (23), where \(f_1(z)\) be in the class \(\overline{H}(\varphi, \psi; t, \beta)\) and \(f_2(z)\) be in the class \(\overline{H}(\varphi, \psi; t, \delta)\). We wish to show that the coefficients of \((f_1 \ast f_2)(z)\) satisfy the required condition given in (10). For \(f_2 \in \overline{H}(\varphi, \psi; t, \delta)\), we note that \(|a_{k,2}| < 1\) and \(|b_{k,2}| < 1\). Now for the convolution functions \((f_1 \ast f_2)(z)\), we obtain

\[
\sum_{k=2}^{\infty} \frac{\lambda_k}{k} \left( \frac{k - \beta}{\delta - 1} \right) |a_{k,1}| |a_{k,2}| + \sum_{k=1}^{\infty} \frac{\mu_k}{k} \left( \frac{k + \beta}{\delta - 1} \right) |b_{k,1}| |b_{k,2}|
\]

\[
\leq \sum_{k=2}^{\infty} \frac{\lambda_k}{k} \left( \frac{k - \beta}{\delta - 1} \right) |a_{k,1}| + \sum_{k=1}^{\infty} \frac{\mu_k}{k} \left( \frac{k + \beta}{\delta - 1} \right) |b_{k,1}|
\]

\[
\leq \sum_{k=2}^{\infty} \frac{\lambda_k}{k} \left( \frac{k - \beta}{\delta - 1} \right) |a_{k,1}| + \sum_{k=1}^{\infty} \frac{\mu_k}{k} \left( \frac{k + \beta}{\delta - 1} \right) |b_{k,1}| \leq 1,
\]

since \(1 < \beta \leq \delta \leq \frac{4}{3}\) and \(f_1 \in \overline{H}(\varphi, \psi; t, \beta)\). Thus

\[
(f_1 \ast f_2)(z) \in \overline{H}(\varphi, \psi; t, \beta) \subset \overline{H}(\varphi, \psi; t, \delta).
\]

The proof of the assertion (27) is similar, thus, we omit it. This completes the proof of Theorem.

Next we show that \(\overline{H}(\varphi, \psi; t, \beta)\) is closed under convex combinations of its members.

Theorem 6. The class \(\overline{H}(\varphi, \psi; t, \beta)\) is closed under convex combination.

Proof. For \(i = 1, 2, \ldots\), let \(f_i \in \overline{H}(\varphi, \psi; t, \beta)\), where

\[
f_i(z) = z + \sum_{k=2}^{\infty} |a_{k,i}| z^k - \sum_{k=1}^{\infty} |b_{k,i}| z^k (z \in U; i = 1, 2, \ldots),
\]

(28)

then from (10), for \(\sum_{i=1}^{\infty} m_i = 1, 0 \leq m_i < 1\), the convex combination of \(f_i\) may be written as

\[
\sum_{i=1}^{\infty} m_i f_i(z) = z + \sum_{k=2}^{\infty} \left( \sum_{i=1}^{\infty} m_i |a_{k,i}| \right) z^k - \sum_{k=1}^{\infty} \left( \sum_{i=1}^{\infty} m_i |b_{k,i}| \right) z^k.
\]

(29)

Then by (29), we have

\[
\sum_{k=2}^{\infty} \frac{\lambda_k}{k} \left( \frac{k - \beta}{\delta - 1} \right) \left( \sum_{i=1}^{\infty} m_i |a_{k,i}| \right) + \sum_{k=1}^{\infty} \frac{\mu_k}{k} \left( \frac{k + \beta}{\delta - 1} \right) \left( \sum_{i=1}^{\infty} m_i |b_{k,i}| \right) \leq 1.
\]
\[
\sum_{i=1}^{\infty} m_i \left[ \sum_{k=2}^{\infty} \frac{\lambda_k}{k} \left( \frac{k-t\beta}{\beta-1} \right) |a_{k,i}| + \frac{\mu_k}{k} \left( \frac{k+t\beta}{\beta-1} \right) |b_{k,i}| \right] \\
\leq \sum_{i=1}^{\infty} m_i \leq 1.
\]

This completes the proof of Theorem. \(\square\)

## 5. Integral Operator

In this section we examine a closure property of the class \(\overline{M}_{\phi, \psi}(\varphi, \psi; t, \beta)\) under the generalized Bernardi-Libera-Livingston integral operator (see [5, 12]) \(L_c(f(z))\) which is defined by

\[
L_c(f(z)) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt, \quad c > -1.
\]  

**Theorem 7.** Let \(f(z) \in \overline{M}_{\phi, \psi}(\varphi, \psi; t, \beta)\). Then \(L_c(f(z)) \in \overline{M}_{\phi, \psi}(\varphi, \psi; t, \beta)\).

**Proof.** From (30), it follows that

\[
L_c(f(z)) = \frac{c+1}{z^c} \left[ \int_0^z t^{c-1} \left( h(t) + g(t) \right) dt \right]
\]

\[
= \frac{c+1}{z^c} \left[ \int_0^z t^{c-1} \left( t + \sum_{k=2}^{\infty} a_k t^k \right) dt - \int_0^z \left( t^{c-1} \sum_{k=1}^{\infty} b_k t^k \right) dt \right]
\]

\[
= z + \sum_{k=2}^{\infty} A_k z^k - \sum_{k=1}^{\infty} B_k z^k,
\]

where

\[
A_k = \left( \frac{c+1}{c+k} \right) a_k, \quad B_k = \left( \frac{c+1}{c+k} \right) b_k.
\]

Therefore,

\[
\sum_{k=2}^{\infty} \frac{\lambda_k}{k} \left( \frac{k-t\beta}{\beta-1} \right) \left( \frac{c+1}{c+k} \right) |a_k| + \sum_{k=1}^{\infty} \frac{\mu_k}{k} \left( \frac{k+t\beta}{\beta-1} \right) \left( \frac{c+1}{c+k} \right) |b_k|
\]

\[
\leq \sum_{k=2}^{\infty} \frac{\lambda_k}{k} \left( \frac{k-t\beta}{\beta-1} \right) |a_k| + \sum_{k=1}^{\infty} \frac{\mu_k}{k} \left( \frac{k+t\beta}{\beta-1} \right) |b_k| \leq 1.
\]

Since \(f(z) \in \overline{M}_{\phi, \psi}(\varphi, \psi; t, \beta)\), by using Theorem 1, then \(L_c(f(z)) \in \overline{M}_{\phi, \psi}(\varphi, \psi; t, \beta)\). This completes the proof of Theorem. \(\square\)

For suitable choice of \(h(z)\) and \(g(z)\) we can obtain the following remarks.
Remark 2.

i) Putting $\varphi = \psi = \frac{z}{(1-z)^r}$ and $t = \sigma = 1$ in the above results, we obtain the corresponding results obtained by Porwal [15];

ii) Putting $\varphi = \psi = z + \sum_{k=2}^{\infty} k\Gamma_k(\alpha_k)z^k$ and $t = \sigma = 1$ in the above results, we obtain the corresponding results obtained by Pathak et al. [14];

iii) Putting $\varphi = \psi = \frac{z^2}{(1-z)^2}$, $t = 1$ and $\sigma = -1$ in the above results, we obtain new results of the class $\mathcal{NH}(\beta)$;

iv) Putting $\varphi = \psi = z + \sum_{k=2}^{\infty} k^{n+1}z^k$, $t = 1$, $n \in \mathbb{N}_0$ and $\sigma = (-1)^n$ in the above results, we obtain new results of the class $\mathcal{M}_{\mathcal{NH}}(\beta, n)$;

v) Putting $\varphi = \psi = z + \sum_{k=2}^{\infty} k^{-n}z^k$, $t = 1$, $n \in \mathbb{N}_0$ and $\sigma = (-1)^{n+1}$ in the above results, we obtain new results of the class $\mathcal{L}_{\mathcal{NH}}(\beta, n)$;

vi) Putting $\varphi = \psi = z + \sum_{k=2}^{\infty} k[1 + \lambda(k-1)]^{-n}z^k$, $t = 1$, $\lambda \geq 0$, $n \in \mathbb{N}_0$ and $\sigma = (-1)^n$ in the above results, we obtain new results of the class $\mathcal{M}_{\mathcal{NH}}(\beta, n, \lambda)$;

vii) Putting $\varphi = \psi = z + \sum_{k=2}^{\infty} k[1 + \lambda(k-1)]^{-n}z^k$, $t = 1$, $\lambda \geq 0$, $n \in \mathbb{N}_0$ and $\sigma = (-1)^n$ in the above results, we obtain new results of the class $\mathcal{L}_{\mathcal{NH}}(\beta, n, \lambda)$;

viii) Putting $\varphi = \psi = z + \sum_{k=2}^{\infty} k\left(\frac{1+\ell+\lambda(k-1)}{1+\ell}\right)^m z^k$, $t = 1$, $\ell, \lambda \geq 0$, $m \in \mathbb{N}_0$ and $\sigma = (-1)^m$ in the above results, we obtain new results of the class $\mathcal{M}_{\mathcal{NH}}(\beta, m, \ell, \lambda)$.

References


