Generalizations of Hadamard Product of Certain Meromorphic Multivalent Functions with Positive Coefficients

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Abstract. In this paper, we used the generalization of the modified-Hadamard products for p-valent meromorphic functions to obtain some results for the classes \( \sum_{\text{p}} S_n^*(\alpha, \beta) \) and \( \sum_{\text{p}} K_n(\alpha, \beta) \), which represent the classes of meromorphically p-valent starlike of order \( \alpha \) and type \( \beta \) and meromorphically p-valent convex of order \( \alpha \) and type \( \beta \) respectively.

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1. Introduction

Let \( \sum_{\text{p,n}} \) denote the class of functions of the form:

\[
\sum_{\text{p,n}} f(z) = \frac{1}{z^p} + \sum_{k=n}^{\infty} a_k z^k \quad (a_k \geq 0; n \geq p; p \in \mathbb{N} = \{1, 2, 3, \ldots\}) \tag{1}
\]

that are analytic and p-valent in the punctured disk 
\( U^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = U \setminus \{0\} \quad (U = \{z \in \mathbb{C} : |z| < 1\}) \).

A function \( f(z) \in \sum_{\text{p,n}} \) is said to be meromorphically p-valent starlike of order \( \alpha \) if it is satisfying the following (see Aouf and Hossen [3] and Kumar et al. [9]):

\[
\text{Re} \left\{ -\frac{zf'(z)}{f(z)} \right\} > \alpha \quad (0 \leq \alpha < p; z \in U^*), \tag{2}
\]

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a function \( f(z) \in \sum_{p,n} \) is said to be meromorphically \( p \)-valent convex of order \( \alpha \) if it is satisfying the following (see Nunokawa and Ahuja [14]):

\[
\text{Re} \left\{ - \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\} > \alpha \quad (0 \leq \alpha < p; z \in U^*). \tag{3}
\]

We denote by \( \sum_{p} S_n^*(\alpha) \) and \( \sum_{p} K_n(\alpha) \) the classes of meromorphically \( p \)-valent starlike of order \( \alpha \) and meromorphically \( p \)-valent convex of order \( \alpha \) respectively, we note that

\[
f(z) \in \sum_{p} K_n(\alpha) \iff -\frac{zf'(z)}{p} \in \sum_{p} S_n^*(\alpha). \tag{4}
\]

We note that the classes \( \sum_{1} S_1^*(\alpha) = \sum S^*(\alpha) \) and \( \sum_{1} K_1(\alpha) = \sum K(\alpha) \) are the classes of meromorphically univalent starlike functions of order \( \alpha \) and meromorphically univalent convex functions of order \( \alpha \) respectively, which have been extensively studied by Pommerenke [15], Clunie [6], Royster [16], Miller [10], Juneja and Reddy [8] and Mogra [12] and others.

Moreover a function \( f(z) \in \sum_{p,n} \) is said to be meromorphically \( p \)-valent starlike of order \( \alpha \) and type \( \beta \) if it is satisfying the following inequality (see Aouf [1] and Mogra [11]):

\[
\left| \frac{zf'(z)}{f(z)} + p \right| < \beta \quad (0 \leq \alpha < p; 0 < \beta \leq 1; z \in U^*), \tag{5}
\]

also a function \( f(z) \in \sum_{p,n} \) is said to be meromorphically \( p \)-valent convex of order \( \alpha \) and type \( \beta \) if it is satisfying the following inequality:

\[
\left| \frac{1 + zf''(z)}{f'(z)} + p \right| < \beta \quad (0 \leq \alpha < p; 0 < \beta \leq 1; z \in U^*). \tag{6}
\]

We denote by \( \sum_{p} S_n^*(\alpha, \beta) \) and \( \sum_{p} K_n(\alpha, \beta) \) the classes of meromorphically \( p \)-valent starlike of order \( \alpha \) and type \( \beta \) and meromorphically \( p \)-valent convex of order \( \alpha \) and type \( \beta \) respectively, we note that

\[
f(z) \in \sum_{p} K_n(\alpha, \beta) \iff -\frac{zf'(z)}{p} \in \sum_{p} S_n^*(\alpha, \beta). \tag{7}
\]

We note that the class \( \sum_{1} S_1^*(\alpha, \beta) = \sum S^*(\alpha, \beta) \) was introduced and studied by Mogra et al. [13].
For the functions
\[ f_j(z) = \frac{1}{z^p} + \sum_{k=n}^{\infty} a_{k,j} z^k \quad (a_{k,j} \geq 0; j = 1, 2; n \geq p; p \in \mathbb{N}), \tag{8} \]
we denote by \((f_1 \ast f_2)(z)\) the Hadamard product (or convolution) of the functions \(f_1(z)\) and \(f_2(z)\), that is,
\[ (f_1 \ast f_2)(z) = \frac{1}{z^p} + \sum_{k=n}^{\infty} a_{k,1} a_{k,2} z^k. \tag{9} \]

For any real numbers \(r\) and \(s\), the generalized Hadamard product \((f_1 \Delta f_2)(r, s; z)\) is given by
\[ (f_1 \Delta f_2)(r, s; z) = \frac{1}{z^p} + \sum_{k=n}^{\infty} (a_{k,1})^r (a_{k,2})^s z^k. \tag{10} \]

If we take \(r = s = 1\), then we have
\[ (f_1 \Delta f_2)(1, 1; z) = (f_1 \ast f_2)(z) \quad (z \in U^*). \tag{11} \]

In the present paper, applying methods used by Choi et al. [5], Aouf and Silverman [4] and Darwish and Aouf [7], we will obtain several results for the generalized Hadamard product of functions in the classes \(\sum_p S^*_n(\alpha, \beta)\) and \(\sum_p K_n(\alpha, \beta)\).

### 2. Main Results

Unless otherwise mentioned, we assume in the reminder of this paper that \(0 \leq \alpha < p, 0 < \beta \leq 1, n \geq p, p \in \mathbb{N}\) and \(z \in U^*\).

In order to prove our results for functions belonging to the classes \(\sum_p S^*_n(\alpha, \beta)\) and \(\sum_p K_n(\alpha, \beta)\), we shall need the following lemmas given by Aouf [1, 2] see also Mogra [11].

**Lemma 1.** Let the function \(f(z)\) be defined by (1). Then \(f(z)\) is in the class \(\sum_p S^*_n(\alpha, \beta)\) if and only if
\[ \sum_{k=n}^{\infty} [(k + p) + \beta(k + 2\alpha - p)] a_k \leq 2\beta (p - \alpha). \tag{12} \]

**Lemma 2.** Let the function \(f(z)\) be defined by (1). Then \(f(z)\) is in the class \(\sum_p K_n(\alpha, \beta)\) if and only if
\[ \sum_{k=n}^{\infty} \frac{k}{p} [(k + p) + \beta(k + 2\alpha - p)] a_k \leq 2\beta (p - \alpha). \tag{13} \]

Applying Lemma 1 and Lemma 2, we derive:

**Theorem 1.** If the functions \(f_j(z) \ (j = 1, 2)\) defined by (8) are in the classes \(\sum_p S^*_n(\alpha_j, \beta)\) for each \(j\), then
\[ (f_1 \Delta f_2)(\frac{1}{r}, \frac{-1}{r}; z) \in \sum_p S^*_n(\gamma, \beta), \tag{14} \]
where $r > 1$ and

$$\gamma = \min_{k \geq n} \left( p - \frac{(k + p)(1 + \beta)}{2\beta \left[ 1 + \left( \frac{k + p + \beta(k+2\alpha_1-p)}{2\beta(p-\alpha_1)} \right)^{\frac{1}{r}} \left( \frac{k + p + \beta(k+2\alpha_2-p)}{2\beta(p-\alpha_2)} \right)^{\frac{r-1}{r}} \right]} \right).$$

**Proof.** Since $f_j(z) \in \sum_{n} S_n^*(\alpha_j, \beta)$ ($j = 1, 2$), by using Lemma 1, we have

$$\sum_{k=n}^{\infty} \left( \frac{(k + p) + \beta(k + 2\alpha_j - p)}{2\beta(p - \alpha_j)} \right) a_{k,j} \leq 1 \quad (j = 1, 2).$$

Moreover,

$$\left( \sum_{k=n}^{\infty} \left( \frac{(k + p) + \beta(k + 2\alpha_1 - p)}{2\beta(p - \alpha_1)} \right) a_{k,1} \right)^{\frac{1}{r}} \leq 1,$n

and

$$\left( \sum_{k=n}^{\infty} \left( \frac{(k + p) + \beta(k + 2\alpha_2 - p)}{2\beta(p - \alpha_2)} \right) a_{k,2} \right)^{\frac{r-1}{r}} \leq 1.$$

By using Holder inequality, we get

$$\sum_{k=n}^{\infty} \left( \frac{(k + p) + \beta(k + 2\alpha_1 - p)}{2\beta(p - \alpha_1)} \right)^{\frac{1}{r}} \left( \frac{(k + p) + \beta(k + 2\alpha_2 - p)}{2\beta(p - \alpha_2)} \right)^{\frac{r-1}{r}} a_{k,1} a_{k,2} \leq 1.$$

Since

$$(f_1 \Delta f_2) \left( \frac{1}{r}, \frac{r-1}{r}; z \right) = \frac{1}{z^p} + \sum_{k=n}^{\infty} \left( a_{k,1} \right)^{\frac{1}{r}} \left( a_{k,2} \right)^{\frac{r-1}{r}} z^k,$$

we see that

$$\sum_{k=n}^{\infty} \left( \frac{(k + p) + \beta(k + 2\gamma - p)}{2\beta(p - \gamma)} \right) \left( a_{k,1} \right)^{\frac{1}{r}} \left( a_{k,2} \right)^{\frac{r-1}{r}} \leq 1$$

with

$$\gamma = \min_{k \geq n} \left( p - \frac{(k + p)(1 + \beta)}{2\beta \left[ 1 + \left( \frac{k + p + \beta(k+2\alpha_1-p)}{2\beta(p-\alpha_1)} \right)^{\frac{1}{r}} \left( \frac{k + p + \beta(k+2\alpha_2-p)}{2\beta(p-\alpha_2)} \right)^{\frac{r-1}{r}} \right]} \right).$$
Thus, by using Lemma 1, the proof of Theorem 1 is completed.

**Corollary 1.** If the functions \( f_j(z) \) \((j = 1, 2) \) defined by (8) are in the class \( \sum_p S_n^*(\alpha, \beta) \) for each \( j \), then
\[
(f_1 \Delta f_2) \left( \frac{1}{r}, \frac{r - 1}{r}; z \right) \in \sum_p S_n^*(\alpha, \beta) \quad (r > 1).
\]

**Proof.** In view of Lemma 1, Corollary 1 follows immediately from Theorem 1 by taking \( \alpha_j = \alpha \) \((j = 1, 2)\).

**Theorem 2.** Let the functions \( f_j(z) \) \((j = 1, 2) \) defined by (8) are in the classes \( \sum_p K_n(\alpha_j, \beta) \) for each \( j \), then
\[
(f_1 \Delta f_2) \left( \frac{1}{r}, \frac{r - 1}{r}; z \right) \in \sum_p K_n(\gamma, \beta),
\]
where \( r > 1 \) and \( \gamma \) is defined by (15).

**Proof.** Since \( f_j(z) \in \sum_p K_n(\alpha_j, \beta) \) \((j = 1, 2)\), by using Lemma 2, we have
\[
\sum_{k=n}^{\infty} \left( \frac{k}{p} \right) \left( \frac{(k + p) + \beta(k + 2\alpha_j - p)}{2\beta(p - \alpha_j)} \right) a_{k,j} \leq 1 \quad (j = 1, 2).
\]
Thus the proof of Theorem 2 is similar to that of Theorem 1 where Lemma 2 is used instead of Lemma 1.

**Corollary 2.** If the functions \( f_j(z) \) \((j = 1, 2) \) defined by (8) are in the class \( \sum_p K_n(\alpha, \beta) \) for each \( j \), then
\[
(f_1 \Delta f_2) \left( \frac{1}{r}, \frac{r - 1}{r}; z \right) \in \sum_p K_n(\alpha, \beta) \quad (r > 1).
\]

**Proof.** In view of Lemma 2, Corollary 2 follows immediately from Theorem 2 by taking \( \alpha_j = \alpha \) \((j = 1, 2)\).

**Theorem 3.** Let the functions \( f_j(z) \) \((j = 1, 2, \ldots, m) \) defined by (8) are in the classes \( \sum_p S_n^*(\alpha_j, \beta) \) for each \( j \), and let the function \( F_m(z) \) defined by
\[
F_m(z) = \frac{1}{z^p} + \sum_{k=n}^{\infty} \left( \sum_{j=1}^{m} \left( a_{k,j} \right)^r \right) z^k \quad (z \in U^*; r \geq 2).
\]
Then \( F_m(z) \in \sum_p S_n^*(\gamma_m, \beta) \) \((z \in U)\), where
\[
\gamma_m \leq p - \frac{m(1 + \beta)(n + p) [2\beta(p - \alpha)]^r}{2m^2 \beta [2\beta(p - \alpha)]^r + 2\beta(n + p + \beta(n + 2\alpha - p))^r},
\]
where
\[
\alpha = \min_{1 \leq j \leq m} \{ \alpha_j \}
\]
and
\[
[m(1 + \beta)(n + p) - 2\beta pm][2\beta(p - \alpha)]^r \leq 2\beta p(n + p + \beta(n + 2\alpha - p))^r
\]
Proof. Since $f_j(z) \in \sum p S^*_n(\alpha_j, \beta)$, by using Lemma 1, we obtain
\[
\sum_{k=n}^{\infty} \left( \frac{(k + p) + \beta(k + 2\alpha_j - p)}{2\beta (p - \alpha_j)} \right) a_{k,j} \leq 1 \quad (j = 1, 2, \ldots, m), \tag{30}
\]
and
\[
\sum_{k=n}^{\infty} \left( \frac{(k + p) + \beta(k + 2\alpha_j - p)}{2\beta (p - \alpha_j)} \right)^r \left( \frac{a_{k,j}}{2\beta (p - \alpha_j)} \right) \leq \left( \sum_{k=n}^{\infty} \left( \frac{(k + p) + \beta(k + 2\alpha_j - p)}{2\beta (p - \alpha_j)} \right) a_{k,j} \right)^r \leq 1. \tag{31}
\]
It follows from (31) that
\[
\sum_{k=n}^{\infty} \left( \frac{1}{m} \sum_{j=1}^{m} \left( \frac{(k + p) + \beta(k + 2\alpha_j - p)}{2\beta (p - \alpha_j)} \right)^r \left( \frac{a_{k,j}}{2\beta (p - \alpha_j)} \right) \right) \leq 1. \tag{32}
\]
Putting
\[
\alpha = \min_{1 \leq j \leq m} \{ \alpha_j \},
\]
and by virtue of (32), we find that
\[
\sum_{k=n}^{\infty} \left( \frac{(k + p) + \beta(k + 2\gamma_m - p)}{2\beta (p - \gamma_m)} \right) \sum_{j=1}^{m} (a_{k,j})^r \leq \sum_{k=n}^{\infty} \left( \frac{1}{m} \sum_{j=1}^{m} \left( \frac{(k + p) + \beta(k + 2\alpha_j - p)}{2\beta (p - \alpha_j)} \right)^r \left( \frac{a_{k,j}}{2\beta (p - \alpha_j)} \right) \right) \leq 1 \tag{33}
\]
if
\[
\gamma_m \leq p - \frac{m (1 + \beta) (k + 1) \left[ 2\beta (p - \alpha) \right]^r}{2m\beta \left[ 2\beta (p - \alpha) \right]^r + 2\beta \left[ k + p + \beta (k + 2\alpha - p) \right]^r} \quad (k \geq n). \tag{34}
\]
Now let
\[
g(k) = p - \frac{m (1 + \beta) (k + 1) \left[ 2\beta (p - \alpha) \right]^r}{2m\beta \left[ 2\beta (p - \alpha) \right]^r + 2\beta \left[ k + p + \beta (k + 2\alpha - p) \right]^r} \quad (k \geq n).
\]
Then
\[
g'(k) = \frac{2\beta m (1 + \beta) \left[ 2\beta (p - \alpha) \right]^r \left\{ \left[ k + p + \beta (k + 2\alpha - p) \right]^r - \left[ (1 + \beta) (k + p) \cdot \left( r - 1 \right) + 2\beta (p - \alpha) \right]^r \right\} \cdot \left( \frac{\left[ 2\beta (p - \alpha) \right]^r + 2\beta \left[ k + p + \beta (k + 2\alpha - p) \right]^r}{2\beta m (1 + \beta) \left[ 2\beta (p - \alpha) \right]^r + 2\beta \left[ n + p + \beta (n + 2\alpha - p) \right] \left[ k + p + \beta (k + 2\alpha - p) \right]^r} \right\}^2}{2\beta \left[ 2\beta (p - \alpha) \right]^r + 2\beta \left[ k + p + \beta (k + 2\alpha - p) \right]^r}
\]
\[
\]
\[
\]
The result is sharp, the extremal functions are

\[ f_j(z) = \frac{1}{z^p} + \frac{2\beta (p-\alpha)}{(n+p) + \beta(n+2\alpha-p)} z^n \quad (j = 1, 2, \ldots, m). \]
Taking $\beta = p = 1$, $m = 2$ and $n = 1$ in Corollary 3, we obtain the following corollary:

**Corollary 4** ([8, Theorem 10]). Let the functions $f_j(z)$ ($j = 1, 2$) defined by (8) are in the class $\sum S_1^*(\alpha, 1) = \sum S_1^*(\alpha)$ for each $j$, and let the function $F_2(z)$ defined by

$$F_2(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \left( a_{k,1}^2 + a_{k,2}^2 \right) z^k \quad (z \in U^*). \quad (40)$$

Then $F_2(z) \in \sum S^*(s)(z \in U^*)$, where

$$s_2 = 1 - \frac{4(1 - \alpha)^2}{(1 + \alpha)^2 + 2(1 - \alpha)^2}, \quad (41)$$

and

$$3 - 2\sqrt{2} \leq \alpha \leq 1. \quad (42)$$

**Theorem 4.** Let the functions $f_j(z)$ ($j = 1, 2, \ldots, m$) defined by (8) are in the classes $\sum K_n(\alpha, \beta)$ for each $j$, and let the function $F_m(z)$ defined by (26). Then $F_m(z) \in \sum K_n(\mu_m, \beta)$ ($z \in U$), where

$$\mu_m \leq p - \frac{m \left( 1 + \beta \right) \left( n + p \right) [2\beta (p - \alpha)]^r p^{r-1}}{2m\beta [2\beta (p - \alpha)]^r p^{r-1} + 2\beta \left[ n + p + \beta (n + 2\alpha - p) \right]^r n^{r-1}} \quad (r \geq 2), \quad (43)$$

where

$$\alpha = \min_{1 \leq j \leq m} \left\{ \alpha_j \right\}$$

and

$$\left[ m \left( 1 + \beta \right) \left( n + p \right) p^{r-1} - 2\beta mp^r \right] [2\beta (p - \alpha)]^r \leq 2\beta p \left[ n + p + \beta (n + 2\alpha - p) \right]^r n^{r-1}. \quad (44)$$

**Proof.** Since $f_j(z) \in \sum K_n(\alpha_j, \beta)$ ($j = 1, 2, \ldots, m$), using Lemma 2, we obtain

$$\sum_{k=0}^{\infty} \left( \frac{k}{p} \right) \left( \frac{(k + 1) + \beta (k + 2\alpha - p)}{2b (p - \alpha)} \right) a_{k,j} \leq 1 \quad (j = 1, 2, \ldots, m). \quad (45)$$

Thus the proof of Theorem 4 is similar to that of Theorem 3 where Lemma 2 is used instead of Lemma 1, therefore it is omitted.

Taking $r = 2$ and $\alpha_j = \alpha$ ($j = 1, 2, \ldots, m$) in Theorem 4, we obtain the following corollary:

**Corollary 5.** Let the functions $f_j(z)$ ($j = 1, 2, \ldots, m$) defined by (8) are in the class $\sum K_n(\alpha, \beta)$ for each $j$, and let the function $F_m(z)$ defined by (36). Then $F_m(z) \in \sum K_n(\lambda_m, \beta)$ ($z \in U^*$), where

$$\lambda_m = p - \frac{m \left( 1 + \beta \right) \left( n + p \right) [2\beta (p - \alpha)]^2 p}{2m\beta [2\beta (p - \alpha)]^2 p^2 + 2\beta \left[ n + p + \beta (n + 2\alpha - p) \right]^2 n}, \quad (46)$$

and

$$\left[ m \left( 1 + \beta \right) \left( n + p \right) p - 2\beta mp^2 \right] [2\beta (p - \alpha)]^2 \leq 2\beta p \left[ n + p + \beta (n + 2\alpha - p) \right]^2 n. \quad (47)$$
Taking $\beta = p = 1$, $m = 2$ and $n = 1$ in Corollary 5, we obtain the following corollary:

**Corollary 6.** Let the functions $f_j(z)$ ($j = 1, 2$) defined by (8) are in the class $\sum_1 K_1(\alpha, 1) = \sum K(\alpha)$ for each $j$, where $\alpha$ satisfy (42) and let the function $F_2(z)$ defined by (40), then $F_2(z) \in \sum_1 K_1(\delta_2, 1) = \sum K(\delta_2)$, where $\delta_2$ is defined by (41).

**Remark 1.** Putting $\beta = p = 1$ in all the above results, we obtain the results obtained by Aouf and Silverman [4].

**References**


